ReadMe

5th August 2025

Abstract

This file is for the explanation of the code.

Contents

1 Introduction

In total, there are two folders: `MatlabCode` and `MathematicaCode`. The `MatlabCode` folder contains the code for the numerical computation of the new representation of Baik–Rains distribution. The `MathematicaCode` folder provides the code used to compute one of the expressions appearing in the paper, namely the term

$$\det\left(\mathbb{1} - X\hat{R}\right), \tag{1.1}$$

which appears in Section 5.

2 Matlab Code

Let $F_{\mathrm{BR},\tau}(s)$ be the Baik-Rains distribution with parameter $\tau \in \mathbb{R}$. The new representation is given by

$$F_{\mathrm{BR},\tau}(s) = \det(\mathbb{1} - P_{s+\tau^2} K^{\mathrm{BR},\tau} P_{s+\tau^2})_{L^2(\mathbb{R})} \tag{2.1}$$ E:oldAndNew

where $K^{\mathrm{BR},\tau}$ is a 2×2 matrix kernel with entries

$$
\begin{aligned}
K_{11}^{\mathrm{BR},\tau}(\xi,\zeta) &:= K_{\mathrm{Ai}}(\xi,\zeta) - \mathrm{Ai}(\zeta)\int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(\xi+\lambda)e^{-\tau\lambda} + e^{-\frac{\tau^3}{3}+\tau\xi}\mathrm{Ai}(\zeta)\\
K_{22}^{\mathrm{BR},\tau}(\xi,\zeta) &:= K_{\mathrm{Ai}}(\xi,\zeta) - \mathrm{Ai}(\xi)\int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(\zeta+\lambda)e^{\tau\lambda} + e^{\frac{\tau^3}{3}-\tau\zeta}\mathrm{Ai}(\xi)\\
K_{21}^{\mathrm{BR},\tau}(\xi,\zeta) &:= \tau^2 K_{\mathrm{Ai}}(\xi,\zeta) - \tau(\partial_\xi K_{\mathrm{Ai}}(\xi,\zeta) - \partial_\zeta K_{\mathrm{Ai}}(\xi,\zeta)) - \partial_\xi\partial_\zeta K_{\mathrm{Ai}}(\xi,\zeta)\\
K_{12}^{\mathrm{BR},\tau}(\xi,\zeta) &:= -\int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\gamma\; K_{\mathrm{Ai}}(\xi+\gamma,\zeta+\lambda)e^{\tau(\lambda-\gamma)}\\
&\qquad + e^{\frac{\tau^3}{3}-\tau\zeta}\int_0^\infty \mathrm{d}\gamma \mathrm{Ai}(\xi+\gamma)\gamma e^{-\tau\gamma} + e^{-\frac{\tau^3}{3}+\tau\xi}\int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(\zeta+\lambda)\lambda e^{\tau\lambda}\\
&\qquad - e^{\tau(\xi-\zeta)}(\tau^2 - \min\{\xi,\zeta\}),
\end{aligned}
\tag{2.2}
$$

E:brtau

where K_{Ai} is the Airy kernel

$$
K_{\mathrm{Ai}}(\xi,\zeta) := \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(\xi+\lambda)\mathrm{Ai}(\zeta+\lambda). \tag{2.3}
$$

E:airk

The folder `MatlabCode` contains five files. We explain their purposes below.

Remark 2.1. *In order to run the program, one needs to install Bornemann's* ***`RMTFredholmToolbox`*** *package.*

2.1 `F_tau.m`

This file implements the function $F_{\mathrm{BR},\tau}$ defined in (2.2) using Bornemann's method [1]. The function defined in this file has the following form:

```
F_tau(s, tau),
```
(2.4)

where $s,\tau\in\mathbb{R}$ correspond to their counterparts in $F_{\mathrm{BR},\tau}(s)$.

2.2 `K_tau_11.m`

This file implements the function $K_{11}^{\mathrm{BR},\tau}(x,y)$ defined in (2.2). The integral is evaluated via Clenshaw–Curtis method. The function defined in this file has the following form:

```
K_tau_11(x,y,tau),
```
(2.5)

where $x,y,\tau\in\mathbb{R}$ correspond to their counterparts in $K_{11}^{\mathrm{BR},\tau}(x,y)$.

2.3 `K_tau_12.m`

This file implements the function $K_{12}^{\mathrm{BR},\tau}(x,y)$ defined in (2.2). The integral is evaluated via Clenshaw–Curtis method. The function defined in this file has the following form:

```
K_tau_12(x,y,tau),
```
(2.6)

where $x,y,\tau\in\mathbb{R}$ correspond to their counterparts in $K_{12}^{\mathrm{BR},\tau}(x,y)$.

2.4 `K_tau_21.m`

This file implements the function $K_{21}^{\mathrm{BR},\tau}(x,y)$ defined in (2.2). The integral is evaluated via Clenshaw–Curtis method. The function defined in this file has the following form:

$$\texttt{K\_tau\_21(x,y,tau)}, \tag{2.7}$$

where $x, y, \tau \in \mathbb{R}$ correspond to their counterparts in $K_{21}^{\mathrm{BR},\tau}(x,y)$.

2.5 `K_tau_22.m`

This file implements the function $K_{22}^{\mathrm{BR},\tau}(x,y)$ defined in (2.2). The integral is evaluated via Clenshaw–Curtis method. The function defined in this file has the following form:

$$\texttt{K\_tau\_22(x,y,tau)}, \tag{2.8}$$

where $x, y, \tau \in \mathbb{R}$ correspond to their counterparts in $K_{22}^{\mathrm{BR},\tau}(x,y)$.

2.6 `BR_1.txt`

In this file, we provide the sequence $(x_i, F_{\mathrm{BR},1}(x_i))$, where $x_i := -4 + 0.125 \cdot i$ for $i \in \{0, 1, \ldots, 64\}$. The data is computed using (2.1) by applying Bornemann's method [1].

2.7 `BR_1_2.txt`

In this file, we provide the sequence $(x_i, F_{\mathrm{BR},1/2}(x_i))$, where $x_i := -4 + 0.125 \cdot i$ for $i \in \{0, 1, \ldots, 64\}$. The data is computed using (2.1) by applying Bornemann's method [1].

3 Mathematica Code

The folder `MathematicaCode` contains a single file called `Determinant`. In the paper, we show that

$$\begin{aligned} &\det\left(\mathbb{1} - X\hat{R}\right) \\ &= \left(1 - \int_0^\infty \mathrm{d}x\, \phi_1(x+t) - \langle \psi_1 \mid \phi_2 \rangle_t\right)(1 - \langle \psi_2 \mid \phi_3 \rangle_t) - \langle \psi_1 \mid \phi_3 \rangle_t \langle \psi_2 \mid \phi_2 \rangle_t, \end{aligned} \tag{3.1}$$

E:determinant_

with

$$\begin{aligned} &\int_0^\infty \mathrm{d}x \phi_1(x+t) = -tg(t) + G(t) - F(t) \\ &\phi_2(x) = tG(x,t) - tF(t) + \Phi(x,t) + \Psi(x) + \Psi(t) \\ &\phi_3(x) = 1 + F(x), \qquad \psi_1(x) = \psi(x,t), \qquad \psi_2(x) = f(x), \end{aligned} \tag{3.2}$$

E:rewrite

where

$$\begin{aligned} G(x,t) &:= -\int_0^\infty \mathrm{d}\gamma K_{\mathrm{Ai}}(x+\gamma,t), & \Psi(x) &:= \int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\gamma \mathrm{Ai}(x+\lambda+\gamma), \\ \Phi(x,t) &:= -\int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\gamma K_{\mathrm{Ai}}(x+\lambda, t+\gamma), & \psi(x,t) &:= \partial_1 K_{\mathrm{Ai}}(x,t). \end{aligned} \tag{3.3}$$

For two functions a and b, we use

$$\texttt{ab} \tag{3.4}$$

in the code to denote the inner product $\langle a \mid b \rangle_t$. In the special case where $b \equiv 1$, we use the notation `aOne` instead. The expression `a` represents the evaluation $a(t)$. Additionally, we use `intphi1` to denote the integral of the function φ_1 appearing in the definition of $F_{\mathrm{BR},\tau}$.

$$\int_0^\infty \mathrm{d}x \phi_1(x+t). \tag{3.5}$$

With this notation, the Mathematica code for (3.1) is

$$(1 - \mathtt{intphi1} - \mathtt{psi1phi2})(1 - \mathtt{psi2phi3}) - \mathtt{psi1phi3} \cdot \mathtt{psi2phi2} \tag{3.6}$$ e36

With (3.2), we then have

$$\begin{aligned}
\mathtt{intphi1} &= -\mathtt{t} \cdot \mathtt{g} + \mathtt{G} - \mathtt{F} \\
\mathtt{psi1phi2} &= \mathtt{t} \cdot \mathtt{psiG} - \mathtt{t} \cdot \mathtt{F} \cdot \mathtt{psiOne} + \mathtt{psiPhi} + \mathtt{Psi} \cdot \mathtt{psiOne} + \mathtt{psiPsi} \\
\mathtt{psi2phi3} &= \mathtt{fOne} + \mathtt{fF} \\
\mathtt{psi1phi3} &= \mathtt{psiOne} + \mathtt{psiF} \\
\mathtt{psi2phi2} &= \mathtt{t} \cdot \mathtt{fG} - \mathtt{t} \cdot \mathtt{F} \cdot \mathtt{fOne} + \mathtt{fPhi} + \mathtt{Psi} \cdot \mathtt{fOne} + \mathtt{fPsi}
\end{aligned} \tag{3.7}$$ e37

Each term appearing on the right hand side can be expressed in terms of $\mathbf{x}$, $\langle g \mid g \rangle$, $g(t)$ and so on. In particular, we define $\mathbf{y} := \mathbf{x}^{-1}$ and in the Mathematica code we denote $\mathbf{x}$ as `x` and $\mathbf{y}$ as `y`. Using those expression and plugging (3.7) back to (3.7), we obtain an expression:

$$\begin{aligned}
&- \mathtt{FOneg} - \mathtt{FOnegg} - FF(g + \mathtt{gg}) - \mathtt{FH}(\mathtt{g} + \mathtt{gg}) - \mathtt{g} \cdot \mathtt{HOne} - \mathtt{gg} \cdot \mathtt{HOne} + \mathtt{g} \cdot \mathtt{Psi} + \mathtt{gg} \cdot \mathtt{Psi} + \mathtt{g} \cdot \mathtt{t} + \mathtt{gg} \cdot \mathtt{t} \\
&+ (-\mathtt{G} - \mathtt{H} + 1/2\mathtt{F}^2 \cdot \mathtt{g} \cdot \mathtt{t} + \mathtt{g} \cdot \mathtt{G} \cdot \mathtt{t} + 1/2 \cdot \mathtt{F}^2 \cdot \mathtt{gg}t + \mathtt{G} \cdot \mathtt{gg} \cdot \mathtt{t}) \cdot \mathtt{x} + \mathtt{x}^2
\end{aligned} \tag{3.8}$$ efirst

Using now

$$\mathtt{G} = -\mathtt{H} \qquad \mathtt{F}^2/2 = -\mathtt{G} \tag{3.9}$$

The first line of (3.8) is equal to

$$-((\mathtt{g} + \mathtt{gg})(\mathtt{FF} + \mathtt{FH} + \mathtt{FOne} + \mathtt{HOne} - \mathtt{Psi} - \mathtt{t})) \tag{3.10}$$

which then gives the desired result.

References

Born08 [1] F. Bornemann, *On the numerical evaluation of Fredholm determinants*, Math. Comput. **79** (2009), 871–915.